# Rule Removal for Confluence


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joint work with

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**JAIST**



13th IWC, July 9, 2024

# Quiz: Confluent?

TRS

$$\mathsf{f}(x) \xrightarrow{\mathbf{1}} \mathsf{f}(\mathsf{f}(x)) \qquad \mathsf{f}(\mathsf{f}(x)) \xrightarrow{\mathbf{2}} \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(x))))$$

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- are we allowed to remove the rule in confluence analysis?

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**Theorem (Nagele et al. 2015 and Shintani and Hirokawa 2015)**

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**Proof.**

by assumption $\to_{\mathcal{R}}^{*} = \to_{\mathcal{S}}^{*}$, so $\mathcal{R}$ and $\mathcal{S}$ are equi-confluent □

## Example

we show confluence of TRS $\mathcal{R}$:

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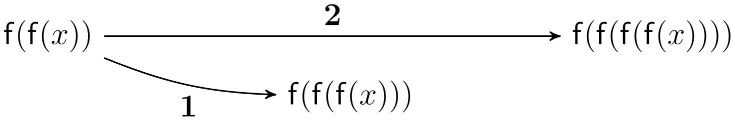

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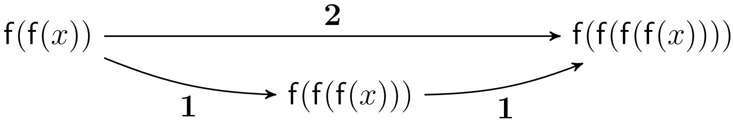

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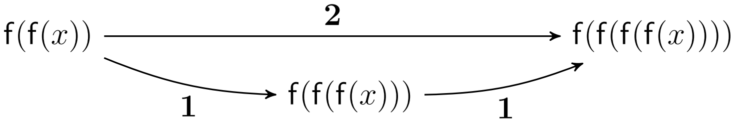


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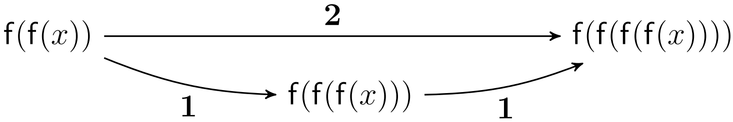


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### Question

are there other rule removal criteria?

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**Question**

any additional condition to ensure "$\mathsf{CR}(\mathcal{R}) \implies \mathsf{CR}(\mathcal{S})$" ?

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3. experiments — joint work with Fuyuki Kawano

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3. confluence is preserved under signature extensions □

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**Corollary**

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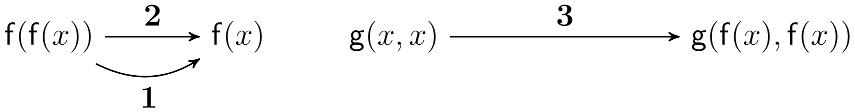

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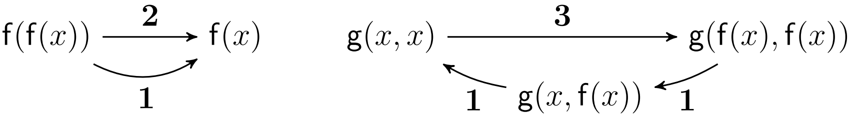

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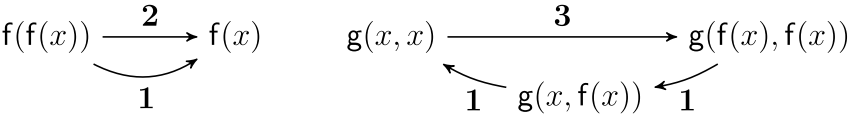


and $\mathcal{F}\mathsf{un}(\{\mathbf{1}\}) = \{\mathsf{f}\}$ leads to

## Example

we show confluence of TRS $\mathcal{R}$:

$$\mathsf{f}(x) \xrightarrow{\mathbf{1}} x \qquad \mathsf{f}(\mathsf{f}(x)) \xrightarrow{\mathbf{2}} \mathsf{f}(x) \qquad \mathsf{g}(x, x) \xrightarrow{\mathbf{3}} \mathsf{g}(\mathsf{f}(x), \mathsf{f}(x))$$

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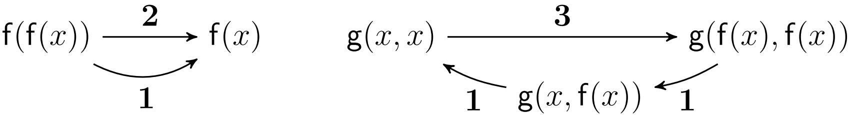


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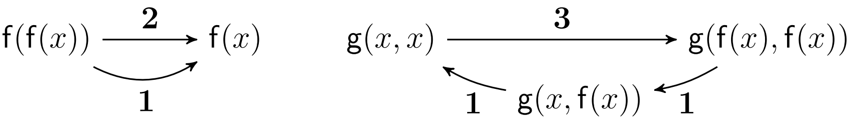


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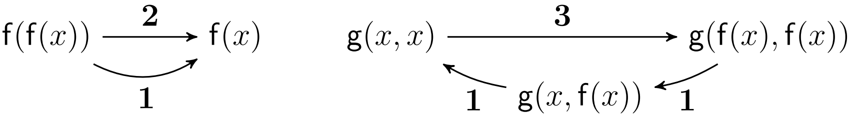


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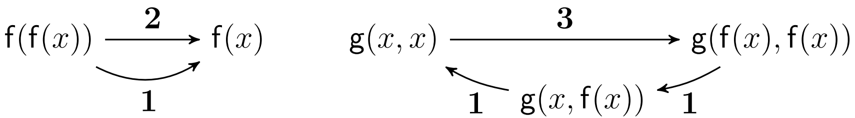


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2 $\mathsf{CR}(\{\mathbf{1}\})$ by Knuth and Bendix' criterion

# Rule Removal by Advanced Criteria

# Parallel Critical Pair Closing Systems

**Theorem (Shintani and Hirokawa 2024)**

$\mathsf{CR}(\mathcal{S}) \implies \mathsf{CR}(\mathcal{R})$ if $\mathcal{R}$ is left-linear, $\mathcal{S} \subseteq \mathcal{R}$, and $\mathsf{PCP}(\mathcal{R}) \subseteq \leftrightarrow_{\mathcal{S}}^{*}$ ($\bigstar$)

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**Corollary**

$\mathsf{CR}(\mathcal{R}) \iff \mathsf{CR}(\mathcal{S})$ if $(\bigstar)$ and $\mathcal{R}\restriction_{\mathcal{S}} \subseteq \to_{\mathcal{S}}^{*}$

## Example

we show confluence of TRS $\mathcal{R}$:

$$\mathsf{S}xyz \xrightarrow{\mathbf{1}} xz(yz) \qquad \mathsf{K}xy \xrightarrow{\mathbf{2}} x \qquad \mathsf{I}x \xrightarrow{\mathbf{3}} x \qquad \mathsf{SKK} \xrightarrow{\mathbf{4}} \mathsf{I}$$

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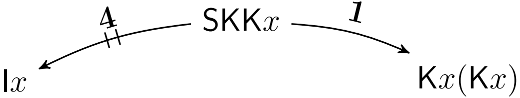

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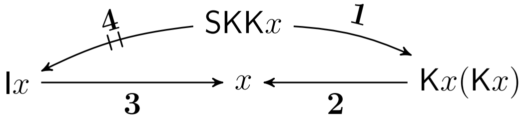

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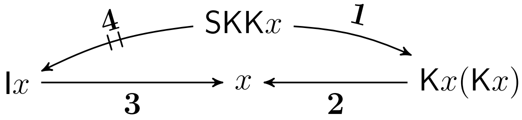


$$\mathcal{F}\mathsf{un}(\mathcal{S}) = \{\circ, \mathsf{K}, \mathsf{I}\}$$

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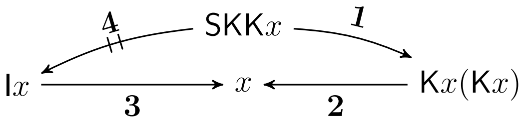


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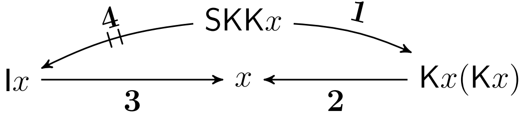


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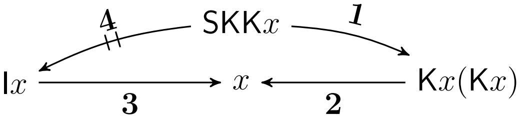


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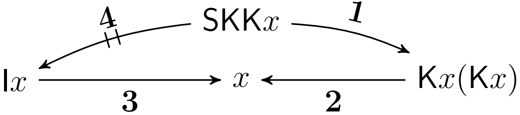


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# Rule Labeling

given rule labeling functions $\phi, \psi : \mathcal{R} \to \mathbb{N}$

- $s \to_{\phi,k} t$ if $s \to_\alpha t$ and $k \geqslant \phi(\alpha)$ for some $\alpha \in \mathcal{R}$
- $s \leftrightarrow_{\curlyvee km} t$ if $s \to_{\psi,i} t$ or $s \;{}_{\phi,i}\!\leftarrow t$ for some $i < k$ or $i < m$

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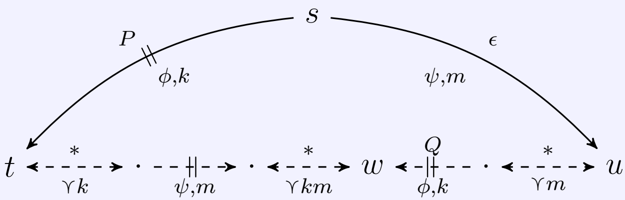
**Definition (parallel version of rule labeling)**

parallel critical peak $t \;{}_{\phi,k}\overset{P}{\leftarrow\!\Vdash} s \xrightarrow{\epsilon}_{\psi,m} u$ is $(\psi, \phi)$-decreasing if




with $\mathcal{V}\mathsf{ar}(w, Q) \subseteq \mathcal{V}\mathsf{ar}(s, P)$

# Rule Labeling

**Theorem (Shintani and Hirokawa 2024)**

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**Theorem (Shintani and Hirokawa 2024)**

CR($\mathcal{S}$) $\implies$ CR($\mathcal{R}$) if following three conditions hold ($\bigstar$):

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2. parallel critical peaks $t \underset{\phi,k}{\leftarrow\!\!\!\mid\!\!\mid\!\!-} s \xrightarrow[\psi,m]{\epsilon} u$ with $k + m \neq 0$ are $(\psi, \phi)$-decreasing

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**Corollary**

CR($\mathcal{R}$) $\iff$ CR($\mathcal{S}$) if ($\bigstar$) and $\mathcal{R}\restriction_{\mathcal{S}} \subseteq \to_{\mathcal{S}}^{*}$

## Example

we show confluence of TRS $\mathcal{R}$:

$$\begin{array}{lll} \mathsf{s}(x) + y \xrightarrow{\mathbf{1}} \mathsf{s}(x+y) & (x+y)+z \xrightarrow{\mathbf{3}} x+(y+z) & \mathsf{d}(x) \xrightarrow{\mathbf{5}} x+x \\ x + \mathsf{s}(y) \xrightarrow{\mathbf{2}} \mathsf{s}(x+y) & \infty \xrightarrow{\mathbf{4}} \mathsf{s}(\infty) & \mathsf{d}(\mathsf{s}(x)) \xrightarrow{\mathbf{6}} \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \end{array}$$

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1 $\mathsf{CR}(\mathcal{R}) \iff \mathsf{CR}(\{\mathbf{1},\mathbf{2},\mathbf{3}\})$ by rule labeling $\mathbf{1},\mathbf{2},\mathbf{3} \mapsto 0$; $\mathbf{4},\mathbf{5} \mapsto 1$; $\mathbf{6} \mapsto 2$

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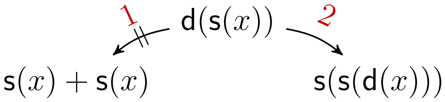

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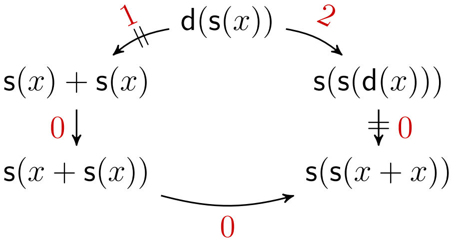

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$$
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\mathsf{s}(x) + y \xrightarrow{\mathbf{1}} \mathsf{s}(x+y) & (x+y)+z \xrightarrow{\mathbf{3}} x+(y+z) & \mathsf{d}(x) \xrightarrow{\mathbf{5}} x+x \\
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\end{array}
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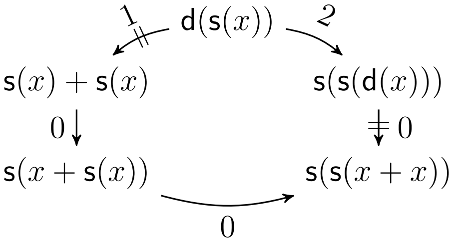


$\mathcal{F}\mathsf{un}(\{\mathbf{1},\mathbf{2},\mathbf{3}\}) = \{\mathsf{s},+\}$

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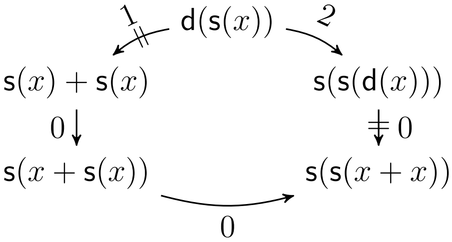


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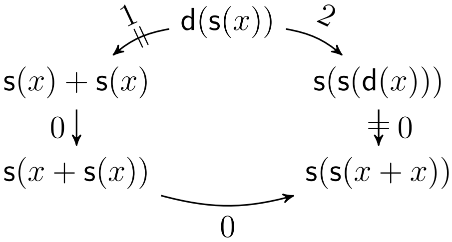


$$\mathcal{F}\mathsf{un}(\{\mathbf{1},\mathbf{2},\mathbf{3}\}) = \{\mathsf{s},+\}$$

$$\mathcal{R}\restriction_{\{\mathbf{1},\mathbf{2},\mathbf{3}\}} = \{\mathbf{1},\mathbf{2},\mathbf{3}\}$$

## Example

we show confluence of TRS $\mathcal{R}$:

$$\mathsf{s}(x) + y \xrightarrow{\mathbf{1}} \mathsf{s}(x+y) \qquad (x+y)+z \xrightarrow{\mathbf{3}} x + (y+z) \qquad \mathsf{d}(x) \xrightarrow{\mathbf{5}} x + x$$

$$x + \mathsf{s}(y) \xrightarrow{\mathbf{2}} \mathsf{s}(x+y) \qquad \infty \xrightarrow{\mathbf{4}} \mathsf{s}(\infty) \qquad \mathsf{d}(\mathsf{s}(x)) \xrightarrow{\mathbf{6}} \mathsf{s}(\mathsf{s}(\mathsf{d}(x)))$$

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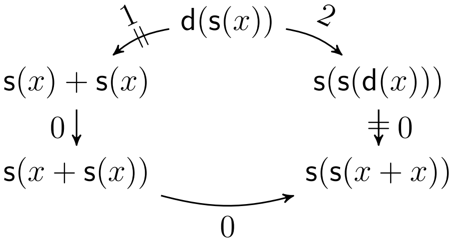


$$\mathcal{F}\mathsf{un}(\{\mathbf{1},\mathbf{2},\mathbf{3}\}) = \{\mathsf{s}, +\}$$

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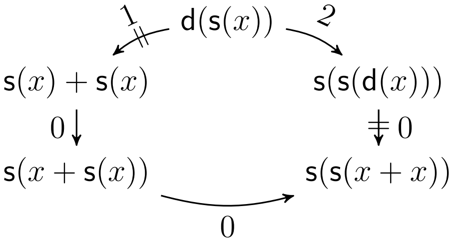


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2 $\mathsf{CR}(\{\mathbf{1},\mathbf{2},\mathbf{3}\})$ by Knuth and Bendix' criterion

# Generalization of Knuth and Bendix' Criterion

**Theorem (Klein and Hirokawa 2012)**

$\mathsf{CR}(\mathcal{S}) \implies \mathsf{CR}(\mathcal{R} \cup \mathcal{S})$ if three following conditions hold ($\bigstar$):

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**Corollary**

$\mathsf{CR}(\mathcal{R} \cup \mathcal{S}) \iff \mathsf{CR}(\mathcal{S})$ if ($\bigstar$) and $(\mathcal{R} \cup \mathcal{S})\restriction_{\mathcal{S}} \subseteq \to_{\mathcal{S}}^{*}$

## Example

we show confluence of TRS

$$\mathsf{eq}(\mathsf{s}(n), x : xs, x : ys) \xrightarrow{\mathbf{1}} \mathsf{eq}(n, xs, ys) \qquad \mathsf{nats} \xrightarrow{\mathbf{3}} \mathsf{0} : \mathsf{inc}(\mathsf{nats})$$

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[2] $\mathsf{CR}(\{\mathbf{3}, \mathbf{4}\})$ by orthogonality

# Experiments by Hakusan

# Experiments on 564 TRSs (ARI Database)

- 60 seconds timeout (PC with Core i5-1340P 1.5GHz)
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# Limitation of $\mathcal{R}{\restriction_{\mathcal{S}}} \subseteq \to_{\mathcal{S}}^*$

consider TRS $\mathcal{R}$ (ARI problem $\#1$):

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**thanks for your attention!**