Second-order Church-Rosser Modulo, Without Normalization

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Consequence The proposition is Permutation $(l_1 ++ l_2, l_2 ++ l_1)$ is <u>ill-typed</u>

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A better solution Make $n_2 + n_1$ and $n_1 + n_2$ convertible So that isPermutation($l_1 ++ l_2, l_2 ++ l_1$) is well-typed

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Goal Church-Rosser rewrite system for the above (second-order) equational theory which can be shown terminating over typed terms

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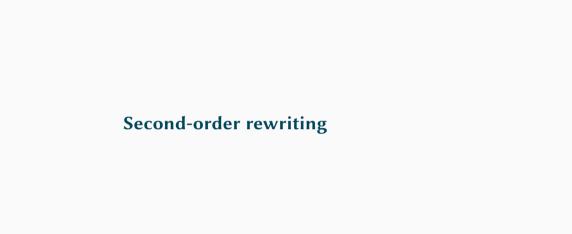
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Each criterion proves CR modulo for a variant of our example



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Example If arity(@) = (0,0) and arity(λ) = (1) then @($\lambda(x.x), y$) $\in \mathcal{T}(\mathcal{F})$

Rewrite system Set of rewrite rules $l \mapsto r$

with l a (fully applied) pattern headed by symbol, and $\mathsf{mv}(r) \subseteq \mathsf{mv}(l)$ and $\mathsf{fv}(l) = \mathsf{fv}(r) = \emptyset$

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Rewrite system modulo Pair $(\mathcal{R}, \mathcal{E})$ with \mathcal{R} rewrite system and \mathcal{E} equational system



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But do we need matching modulo?

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Proposition Suppose that \succ satisfies the diamond property² and that we have an unblocking subset $\mathcal{U} \subseteq \mathcal{A}$. Then $a \ (\succ \cup < \cup \sim)^* \ b$ implies $a \, \succ_{\mathcal{U}}^* \circ \sim \circ \, ^*_{\mathcal{U}} \triangleleft b$

 $^{^2} Think\ of \succ$ as simultaneous/orthogonal/multi-step rewriting

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- $E \simeq \square$ implies $E = \square$
- $E \simeq f(\vec{x}_1.t_1...\vec{x}_k.t_k)$ and $f \in \mathcal{F}_{\mathcal{E}} \cap \mathcal{F}_{\mathcal{R}}$ implies $E = f(\vec{x}_1.u_1...\vec{x}_k.u_k)$ and $t_i \simeq u_i$

Criterion 1 Let $(\mathcal{R}, \mathcal{E})$ be a second-order rewrite system modulo st

- 1. Equations $t_1 \approx t_2 \in \mathcal{E}$ are linear, and we have $mv(t_1) = mv(t_2)$
- 2. Symbols in $\mathcal{F}_{\mathcal{E}}$ have a binding arity of the form $(0,\ldots,0)$
- 3. For every context $E \in \mathcal{T}(\mathcal{F}_{\mathcal{E}})$, there is unblocked $E' \in \mathcal{T}(\mathcal{F}_{\mathcal{E}})$ with $E \simeq E'$
- 4. ${\mathcal R}$ is left-linear and no left-hand side is headed by a symbol in ${\mathcal F}_{\mathcal E}$
- 5. Orthogonal/simultaneous/multi-step rewriting \Longrightarrow with ${\mathcal R}$ satisfies diamond prop.

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 By orthogonality of the rewrite rules

Criterion 1 Let $(\mathcal{R}, \mathcal{E})$ be a second-order rewrite system modulo st

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Proof We show that the set of unblocked terms is an unblocking subset for $(\Longrightarrow, \simeq)$ We do some small adjustments and we obtain the result



Criterion 2 Let $(\mathcal{R} \cup \mathcal{S}, \mathcal{E})$ be a second-order rewrite system modulo st

- 1. $\mathcal{R} \cup \mathcal{S}$ is left-linear
- 2. For all $t \approx u \in \mathcal{E}$, we have t, u linear, headed by symbols, and mv(t) = mv(u)
- 3. \mathcal{R} is confluent
- 4. (S, \mathcal{E}) is strong CR modulo
- 5. No critical pairs between $\mathcal R$ and $\mathcal S \cup \mathcal E^\pm$

Where
$$\mathcal{E}^{\pm} := \mathcal{E} \cup \mathcal{E}^{-1}$$

$$\mathcal{R} =$$

$$@(\lambda(x.\mathsf{t}\{x\}),\mathsf{u}) \longmapsto \mathsf{t}\{\mathsf{u}\}$$

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 @ $(\lambda(x.t\{x\}), u) \longmapsto t\{u\}$ $\mathbb{N}_{rec}(0, p, xy.q\{x, y\}) \longmapsto p$

 $\mathbb{N}_{rec}(S(n), p, xy, q\{x, y\}) \longmapsto q\{n, \mathbb{N}_{rec}(n, p, xy, q\{x, y\})\}$

$$S = t + 0 \longrightarrow t$$
 $t + S(u) \longrightarrow S(t + u)$ $0 + t \longrightarrow t$ $S(u) + t \longrightarrow S(t + u)$

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Therefore, we still need to show the subsystem (S, \mathcal{E}) to be strong CR modulo The point is that (S, \mathcal{E}) might be terminating, allowing application of other criteria

The proof idea

The main tool for proving the criterion is the following well-known result:

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Proof idea for Criterion 2 We know that that $(\mathcal{S}, \mathcal{E})$ is strong CR modulo

 ${\cal R}$ commutes with ${\cal E}^{\scriptscriptstyle \pm}$ and ${\cal S}$ by the above result, and with itself by confluence

We conclude with some easy diagram manipulations



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Thank you for your attention!