On Non-triviality of the Hierarchy of Decreasing Church-Rosser Abstract Rewriting Systems

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Let us recall:

Definition

An **abstract rewriting system** (**ARS**) is a pair $(A, \rightarrow)$, where

- A is a set
- $\rightarrow$ is a binary relation on A (*reduction*)

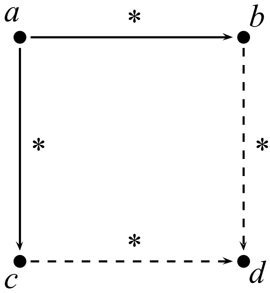
Definition

An ARS $(A, \rightarrow)$ is **terminating**, if there is *no* infinite sequence

$$a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \ldots \text{ (where } a_i \in A\text{)}$$

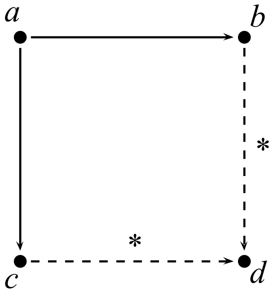
Confluence

$\forall a, b, c \in A\,(a \to^* b \wedge a \to^* c \Rightarrow \exists d \in A\,(b \to^* d \wedge c \to^* d))$



Local confluence

$\forall a, b, c \in A\,(a \to b \wedge a \to c \Rightarrow \exists d \in A\,(b \to^* d \wedge c \to^* d))$



A variant of formulation of **Newman's lemma**:

A **terminating** ARS is **confluent** if(f) it is **locally confluent**

Theorem (see footnote 1)

Let $(A, \to)$ be an ARS that is simultaneously

1. ***countable*** *(the set A is at most countable)*
2. ***acyclic*** *(there is no $a \in A$ such that $a \to^+ a$)*
3. ***strictly inductive*** *(every nonempty chain in the preordered set $(A, \to^*)$ has a least upper bound).*

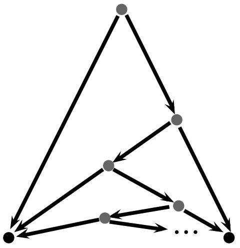
*Then $(A, \to)$ is **confluent** iff it is **locally confluent**.*

Note: if one drops any single precondition among 1-3, the above mentioned statement becomes invalid[1] :

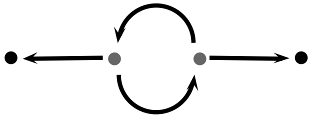
- **Newman's counterexample**:
  countable + acyclic + inductive (but not *strictly inductive*)
- **Hindley's counterexample**:
  (at most) countable + strictly inductive (but not acyclic)
- **"Strengthened Newman's counterexample"** [1]:
  uncountable + acyclic + strictly inductive

[1] I. Ivanov. *On Newman's lemma and non-termination*, CEUR-WS.org vol.3624, pp. 14–24, 2024

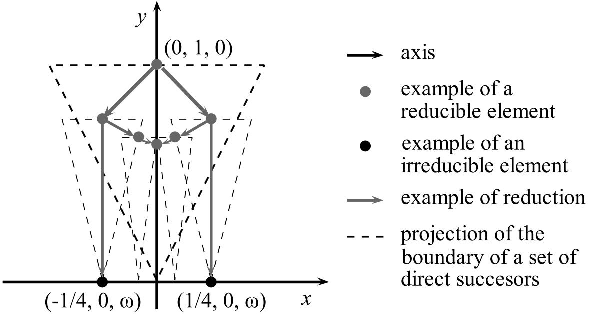
1. Newman's counterexample



2. Hindley's counterexample



3. "Strengthened Newman's counterexample"



Decreasing diagrams

- One can overcome limitations of Newman's lemma using **Van Oostrom's decreasing diagrams method** [2]
- Semi-formally, to prove that an ARS $(A, \rightarrow)$ is confluent:
  1. select a set of **labels** for reduction steps
  2. select a **well-founded partial order** $\prec$ on the set of labels
  3. find a **labeled version** of a $(A, \rightarrow)$ that satisfies a condition reminiscent to the local confluence, but with special constraints on relations between labels of reduction steps.
- Rigorously this can be formulated using the notion of a **decreasing Church-Rosser** (DCR[1]) ARS
- A set of **labels**, an **order** on it, and an **assignment** of labels to rewrite steps can be thought of as **method parameters**

[2] Vincent Van Oostrom. *Confluence by decreasing diagrams*. Theoretical computer science 126, pp. 259–280, 1994

Questions about decreasing diagrams method

- Some general questions (*non-formalized*):
  - *are method parameters **redundant*** ?
  - *how **restrictions** imposed on method parameters influence **capability** of the method to be used to prove confluence* ?
- In part, a setting for studying such questions can be formalized using a **hierarchy of subclasses of DCR ARS** introduced by **J. Endrullis, J.W. Klop, R. Overbeek**[3]

$$DCR_0 \subseteq DCR_1 \subseteq DCR_2 \subseteq ...$$

- semi-formally, DCR_α is the **class of confluent ARS** for which confluence can be proved with the help of the decreasing diagrams method using
  - a **fixed set of labels** $\{\beta \mid \beta < \alpha\}$ (ordinals less than α)
  - and a **fixed order** on them that is a restriction of the usual order on ordinals to $\{\beta \mid \beta < \alpha\}$

[3] J. Endrullis, J.W. Klop, R. Overbeek. *Decreasing diagrams with two labels are complete for confluence of countable systems*, In: FSCD 2018, pp. 14:1–14:15, 2018

Rigorous definition of DCR the hierarchy

Let γ be an ordinal. For any ordinal α denote $\curlyvee\alpha = \{\beta \mid \beta < \alpha\}$

Definition

An ARS $(A, \to)$ **belongs to the class** DCR_γ, if there exists an indexed family $(\to_\alpha)_{\alpha \in (\curlyvee\gamma)}$ of binary relations on A such that $\to = \bigcup_{\alpha<\gamma} \to_\alpha$ and for every ordinals $\alpha, \beta < \gamma$ and for every $a, b, c \in A$, if

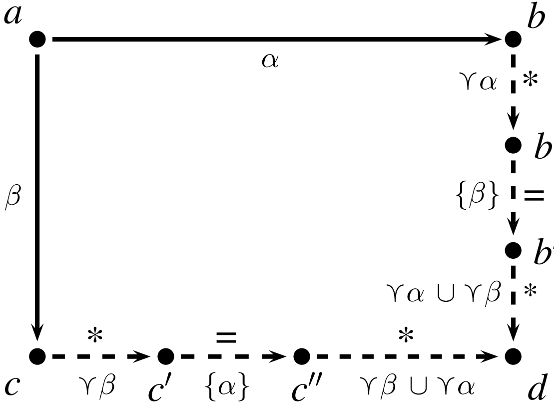
$$a \to_\alpha b \wedge a \to_\beta c,$$

then there exist $b', b'', c', c'', d \in A$ such that

$$\left(b \overset{*}{\underset{\curlyvee\alpha}{\dashrightarrow}} b' \overset{=}{\underset{\{\beta\}}{\dashrightarrow}} b'' \overset{*}{\underset{\curlyvee\alpha\cup\curlyvee\beta}{\dashrightarrow}} d\right) \wedge \left(c \overset{*}{\underset{\curlyvee\beta}{\dashrightarrow}} c' \overset{=}{\underset{\{\alpha\}}{\dashrightarrow}} c'' \overset{*}{\underset{\curlyvee\beta\cup\curlyvee\alpha}{\dashrightarrow}} d\right)$$

Here the following notation is used for any set of ordinals K:

$$\overset{=}{\underset{K}{\dashrightarrow}} = \{(a,a) \mid a \in A\} \cup \bigcup_{\kappa \in K} \to_\kappa \quad \overset{*}{\underset{K}{\dashrightarrow}} = \{(a,a) \mid a \in A\} \cup \left(\bigcup_{\kappa \in K} \to_\kappa\right)^+$$


a
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⋎β ∪ ⋎α
d

Definition

- A subset $B \subseteq A$ is **cofinal** in an ARS $(A, \to)$, if
$$\forall a \in A\ \exists b \in B\ a \to^* b.$$
- An ARS $(A, \to)$ has the **cofinality property**, if for every $a \in A$ there exists a *finite or infinite* reduction sequence $b_0 \to b_1 \to b_2 \to \ldots$ with $b_0 = a$ such that $\{b_0, b_1, b_2, \ldots\}$ is cofinal in $(X, \to \cap (X \times X))$, where $X = \{b \in A \mid a \to^* b\}$.

J. Endrullis, J.W. Klop, R. Overbeek showed[4] that

*Every ARS with the **cofinality property** is in DCR_2.*

For **countable** ARS, confluence $\Leftrightarrow$ cofinality property, so

- confluence of a (confluent) **countable** ARS can **always** be proved with the help of the **decreasing diagrams** method using the **label set** $\{0, 1\}$ ordered in such a way that $0 < 1$.

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[4] J. Endrullis, J.W. Klop, R. Overbeek. *Decreasing diagrams with two labels are complete for confluence of countable systems*, In: FSCD 2018, pp. 14:1–14:15, 2018

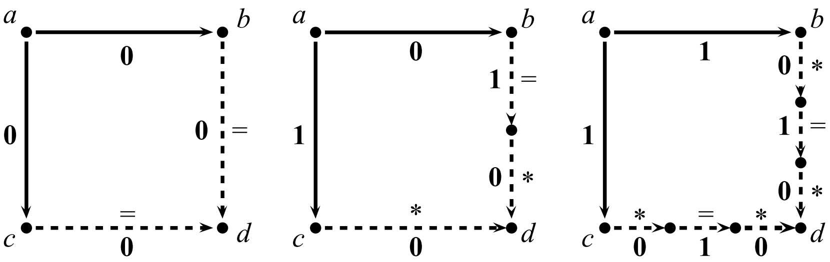
Proposition

An ARS $(A, \to)$ is in DCR_2 ***if and only if*** *there exist binary relations $\to_0, \to_1 \subseteq A \times A$ such that $\to = (\to_0 \cup \to_1)$ and:*

1. $\forall a, b, c \in A\,(a \to_0 b \wedge a \to_0 c \Rightarrow \exists d \in A\,(b \to_0^{=} d \wedge c \to_0^{=} d))$

2. $\forall a, b, c \in A\,(a \to_0 b \wedge a \to_1 c \;\Rightarrow\; \exists b', d \in A\,(b \to_1^{=} b' \wedge b' \to_0^{*} d \wedge c \to_0^{*} d))$

3. $\forall a, b, c \in A\,(a \to_1 b \wedge a \to_1 c \;\Rightarrow\; \exists b', b'', c', c'', d \in A\,(b \to_0^{*} b' \wedge b' \to_1^{=} b'' \wedge b'' \to_0^{*} d \wedge c \to_0^{*} c' \wedge c' \to_1^{=} c'' \wedge c'' \to_0^{*} d))$.



Problems considered in this talk

1. Does there **exist** an (uncountable) confluent ARS **outside** of the class DCR_2 ?

   The obtained answer is YES.

2. Does the DCR hierarchy **collapse** at the level 2 ?

   The obtained answer is NO.

3. How can one **extend** the theorem about the DCR_2 property of ARS with the cofinality property (by J. Endrullis, J.W. Klop, R. Overbeek) ?

   The proposed answer will be described below.

Main results

1. Existence of an ARS in the class $DCR_3 \setminus DCR_2$
2. Cofinal connected subgraph theorem

1. Existence of an ARS in the class $DCR_3 \setminus DCR_2$

Theorem

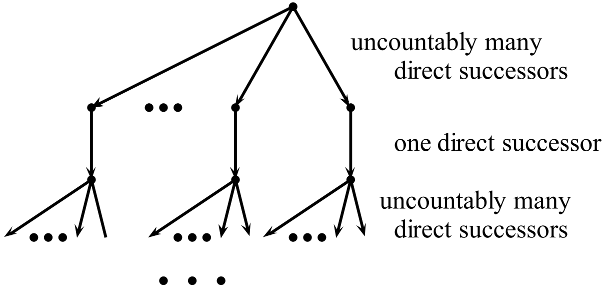
$DCR_3 \setminus DCR_2 \neq \emptyset$

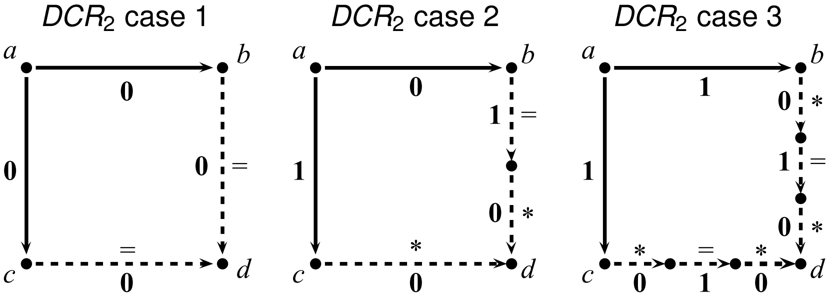
- This result has been formally verified using **Isabelle proof assistant** (using HOL logic).
- Formal proof:

  http://doi.org/10.5281/zenodo.11571490

  (formal theorems `thm_1`, `thm_2`).

Idea of construction of a confluent ARS outside DCR_2





Explicit simplified example

- The original example of an ARS in the class $DCR_3 \backslash DCR_2$ proposed by the author of this talk and formalized in Isabelle can be found in the paper.

- Reviewers **J. Endrullis, F. van Raamsdonk, J.W. Klop** in their review of the initial version of the paper proposed the following ***simplified*** example of a confluent ARS outside DCR_2:

Example

The ARS $(\{1,2,3\} \times \mathcal{P}_{fin}(\mathbb{R}), \to)$, where $\mathcal{P}_{fin}(\mathbb{R})$ denotes the set of all finite subsets of $\mathbb{R}$, and $\to$ is defined by the following rules:

$(1, P) \to (2, P \uplus \{p, q\})$
$(2, P) \to (3, P)$
$(3, P) \to (2, P \uplus \{p\})$

for all $p, q \in \mathbb{R} \backslash P$ with $p \neq q$.

2. Cofinal connected subgraph theorem

Theorem

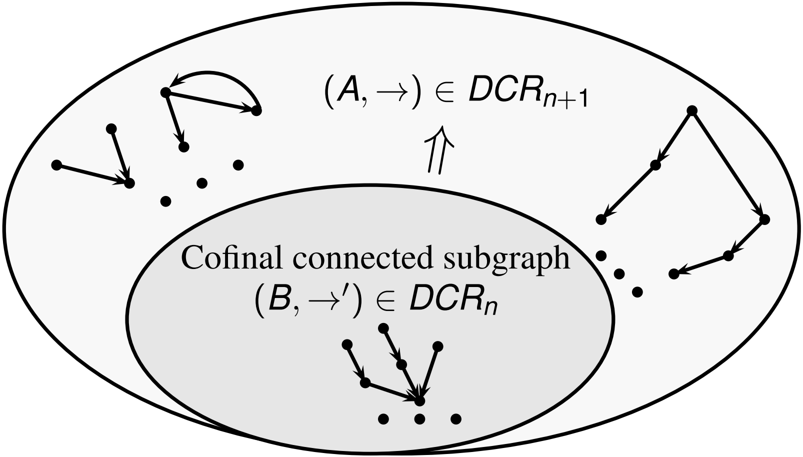
Let $(A, \to)$, $(B, \to')$ be ARS and n be a positive integer. Assume that:

1. $(B, \to')$ *is a (not necessarily induced)* ***subgraph*** *of* $(A, \to)$
2. $(B, \to')$ *is a* ***weakly connected*** *directed graph*
3. B *is* ***cofinal*** *in the ARS* $(A, \to)$*, i.e.* $\forall a \in A\, \exists b \in B\; a \to^* b$.

Then if $(B, \to') \in DCR_n$*, then* $(A, \to) \in DCR_{n+1}$.

- E.g., if $(A, \to)$ has a **cofinal reduction sequence** $b_0 \to b_1 \to$ ...., one can take $(B, \to')$ to be this sequence: $(B, \to')$ **is trivially in** DCR_1, so $(A, \to)$ **must be in** DCR_2**.**
- This result has been formally verified using **Isabelle proof assistant.**

2. Cofinal connected subgraph theorem



Further work

- Further investigation of properties of the **DCR hierarchy**
- Generalization of the **decreasing diagrams** method
- Search for **new applications** of the rewriting theory