universität innsbruck

supported by FWF project I5943-N

Confluence of Logically Constrained Rewrite Systems

Aart Middeldorp

based on joint work with Jonas Schöpf and Fabian Mitterwallner

University of Innsbruck

Outline

1. **IWC**
2. **Logically Constrained Rewrite Systems**
3. **Critical Pairs**
4. **Undecidability**
5. **Transformation**
6. **Confluence Results**
7. **Final Remarks**

International Workshop on Confluence

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IWC Quiz ❶

which years was CoCo not part of IWC ?

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Example

- term rewrite system (TRS)

$$\begin{aligned} \mathsf{sum}(0) &\to 0 & \mathsf{add}(0,y) &\to y \\ \mathsf{sum}(\mathsf{s}(x)) &\to \mathsf{add}(\mathsf{s}(x),\mathsf{sum}(x)) & \mathsf{add}(\mathsf{s}(x),y) &\to \mathsf{s}(\mathsf{add}(x,y)) \end{aligned}$$

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- $\mathcal{R}_{\mathsf{ca}}$ is set of calculation rules and $\mathcal{R}_{\mathsf{rc}} = \mathcal{R} \cup \mathcal{R}_{\mathsf{ca}}$

Example

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substitution σ **respects** constrained rewrite rule $\rho\colon \ell \to r \ [\varphi]$ if

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notation: $\sigma \vDash \rho$

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① position p in s

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such that $s|_p = \ell\sigma$, $t = s[r\sigma]_p$ and $\sigma \vDash \ell \rightarrow r\ [\varphi]$

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  1. position 1
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  3. substitution $\sigma = \{x_1 \mapsto 3, x_2 \mapsto 1, y \mapsto 2\}$

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Confluence Methods for TRSs

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- decreasing diagrams
- development closed critical pairs
- discrimination pairs
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- orthogonality
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- parallel critical pairs
- redundant rules
- rule labeling
- simultaneous critical pairs
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- strongly closed critical pairs
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  4. $\varphi_1\sigma \wedge \varphi_2\sigma$ is satisfiable

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$$\mathsf{f}(x) \to \mathsf{g}(y) \qquad\qquad \mathsf{g}(y) \to \mathsf{a} \quad [y = y]$$

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Theorem (IJCAR 2024)

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IWC Quiz ❶

which years was CoCo not part of IWC ?

IWC Quiz ❶

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IWC Quiz ❷

who published the second most papers at IWC (excluding CoCo tool papers) ?

Outline

1. IWC
2. Logically Constrained Rewrite Systems
3. Critical Pairs
4. **Undecidability**
5. Transformation
6. Confluence Results
7. Final Remarks

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- PCP instance $P = \{(\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)\}$ with $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N \in \{0, 1\}^+$

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Remark

mapping $[\cdot]$ is bijection between $\mathbb{N}$ and candidate strings over $\{1, \ldots, N\}$, for each $N > 0$

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$$\begin{aligned} \mathsf{alpha}(0) &\to \mathsf{e} & \mathsf{alpha}(n) &\to \alpha_i(\mathsf{alpha}(m)) & [\, N \cdot m + i = n \wedge n > 0 \,] \\ \mathsf{beta}(0) &\to \mathsf{e} & \mathsf{beta}(n) &\to \beta_i(\mathsf{beta}(m)) & [\, N \cdot m + i = n \wedge n > 0 \,] \end{aligned}$$

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- constrained critical pairs

$$\mathsf{test}(\mathsf{alpha}(n), \mathsf{beta}(n)) \approx \mathsf{test}(\mathsf{alpha}(m), \mathsf{beta}(m)) \quad [\, n > 0 \wedge m > 0 \,]$$
$$\alpha_i(\mathsf{alpha}(m)) \approx \alpha_j(\mathsf{alpha}(k)) \quad [\, N \cdot m + i = n \wedge n > 0 \wedge N \cdot k + j = n \,]$$
$$\beta_i(\mathsf{beta}(m)) \approx \beta_j(\mathsf{beta}(k)) \quad [\, N \cdot m + i = n \wedge n > 0 \wedge N \cdot k + j = n \,]$$

Remark

constraints belong to decidable fragment of LIA $\implies$ rewrite relation $\to_{\mathcal{R}_P}$ is computable

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- $\mathsf{test}(\mathsf{alpha}(n), \mathsf{beta}(n)) \to^* \bot$ for at least one $n > 0$ because P is non-trivial

Lemma

$\mathcal{R}_P$ is terminating

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RPO (LCTRS version) with precedence

$$\mathsf{start} > \mathsf{test} > \mathsf{alpha} > \mathsf{beta} > 1 > 0 > \mathsf{e} > \top > \bot$$

and well-founded order $\sqsupset_{\mathsf{Int}}$ on integers

$$x \sqsupset_{\mathsf{Int}} y \iff x > y \text{ and } x \geqslant 0$$

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Corollary

(local) confluence of terminating LCTRSs is undecidable, even if underlying theory is decidable

Outline

1. IWC
2. Logically Constrained Rewrite Systems
3. Critical Pairs
4. Undecidability
5. **Transformation**
6. Confluence Results
7. Final Remarks

Definition (Transformation)

LCTRS $\mathcal{R}$ is transformed into TRS $\overline{\mathcal{R}}$ consisting of

$$\ell\tau \to r\tau$$

for all $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}_{\mathsf{rc}}$ and substitutions τ with $\tau \vDash \rho$ and $\mathcal{D}\mathsf{om}(\tau) = \mathcal{LV}\mathsf{ar}(\rho)$

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Example

LCTRS $\mathcal{R}$

$$\mathsf{sum}(x) \to 0 \quad [x \leqslant 0] \qquad\qquad \mathsf{sum}(x) \to x + \mathsf{sum}(x - 1) \quad [x > 0]$$

is transformed into TRS $\overline{\mathcal{R}}$

$$\mathsf{sum}(0) \to 0 \qquad \mathsf{sum}(-1) \to 0 \qquad \mathsf{sum}(-2) \to 0 \qquad \cdots$$

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$$1 + 3 \to 4 \qquad 2 - 1 \to 1 \qquad 3 > 0 \to \mathsf{true} \qquad 2 \leqslant 0 \to \mathsf{false} \qquad \cdots$$

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Goal

adapt concrete confluence methods for TRSs to LCTRSs via transformation

Theorem

for every critical pair $s \approx t$ of $\overline{\mathcal{R}}$ there exist constrained critical pair $u \approx v\ [\varphi]$ of $\mathcal{R}$ and substitution γ such that $s = u\gamma$, $t = v\gamma$ and $\gamma \vDash \varphi$

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for every constrained critical pair $u \approx v\,[\varphi]$ of $\mathcal{R}$ and substitution σ such that $\sigma \vDash \varphi$

1. $u\sigma = v\sigma$ or
2. there exist critical pair $s \approx t$ of $\overline{\mathcal{R}}$ and substitution δ such that $u\sigma = s\delta$ and $t\sigma = v\delta$

Corollary

$\overline{\mathcal{R}}$ is weakly orthogonal $\iff$ $\mathcal{R}$ is weakly orthogonal

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General Proof Idea

$$s \approx t \in \mathrm{CP}(\overline{\mathcal{R}}) \quad \xRightarrow[\sigma \vDash \varphi]{s = u\sigma \quad t = v\sigma} \quad u \approx v\ [\varphi] \in \mathrm{CCP}(\mathcal{R})$$

$$s \to \bullet \leftarrow t \quad \Longleftarrow \quad u \to \bullet \leftarrow v$$

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- rewrite relation $\overset{\sim}{\to}_{\mathcal{R}}$ on constrained terms is defined as $\sim \cdot \to_{\mathcal{R}} \cdot \sim$

Example

LCTRS $\mathcal{R}$ over theory Ints

$$\mathsf{max}(x,y) \to x \quad [x \geqslant y] \qquad\qquad \mathsf{max}(x,y) \to y \quad [y \geqslant x]$$

constrained rewriting

$$\mathsf{max}(1+x, 3+y) \quad [x > 3 \land y = 1]$$

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IWC Quiz ❶

which years was CoCo not part of IWC ?

IWC Quiz ❷

who published the second most papers at IWC (excluding CoCo tool papers) ?

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who are the most frequent PC members at IWC ?

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Kop & Nishida (FroCoS 2013)

... common analysis techniques for term rewriting extend to LCTRSs without much effort

Confluence Methods for LCTRSs

joinable critical pairs for terminating systems

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- $\tilde{\twoheadrightarrow} \ = \ \sim \cdot \twoheadrightarrow \cdot \sim$

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multi-step rewriting

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  3. $\ell\sigma\ [\varphi] \twoheadrightarrow r\tau\ [\varphi]$ if $\rho\colon \ell \to r\ \ [\psi] \in \mathcal{R}_{\mathsf{rc}}$, $\sigma(x)\ [\varphi] \twoheadrightarrow \tau(x)\ [\varphi]$ for all $x \in \mathcal{D}\mathsf{om}(\sigma)$, $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{LV}\mathsf{ar}(\varphi)$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho)$, φ is satisfiable, and $\varphi \Rightarrow \psi\sigma$ is valid
- $\overset{\sim}{\twoheadrightarrow} \ = \ \sim \cdot \twoheadrightarrow \cdot \sim$
- constrained critical pair $s \approx t\ [\varphi]$ is
  - development closed if $s \approx t\ [\varphi] \overset{\sim}{\twoheadrightarrow}_{\geqslant 1} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$
  - almost development closed if it is no overlay and development closed, or it is overlay and $s \approx t\ [\varphi] \overset{\sim}{\twoheadrightarrow}_{\geqslant 1} \cdot \overset{\sim}{\to}^{*}_{\geqslant 2} u \approx v\ [\psi]$ for some trivial $u \approx v\ [\psi]$

Definitions

- multi-step relation $\twoheadrightarrow$ on constrained terms for LCTRS $\mathcal{R}$ is defined inductively as follows:
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- LCTRS is (almost) development closed if all its critical pairs are (almost) development closed

Example

LCTRS $\mathcal{R}$ over theory Ints

$$\mathsf{f}(x,y) \to \mathsf{h}(\mathsf{g}(y, 2 \cdot 2)) \quad [x \leqslant y \wedge y = 2] \qquad \mathsf{g}(x,y) \to \mathsf{g}(y,x) \qquad \mathsf{h}(x) \to x$$

$$\mathsf{f}(x,y) \to \mathsf{c}(4,x) \quad [y \leqslant x] \qquad \mathsf{c}(x,y) \to \mathsf{g}(4,2) \quad [x \neq y]$$

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$$\mathsf{h}(\mathsf{g}(y, 2 \cdot 2)) \approx \mathsf{c}(4,x) \quad [\varphi] \qquad \mathsf{c}(4,x) \approx \mathsf{h}(\mathsf{g}(y, 2 \cdot 2)) \quad [\varphi]$$

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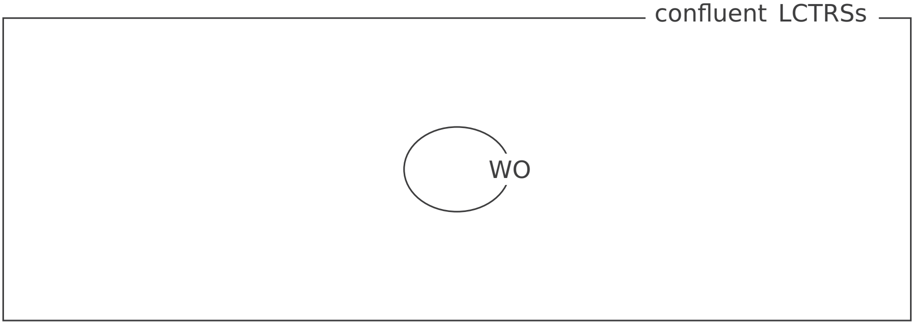
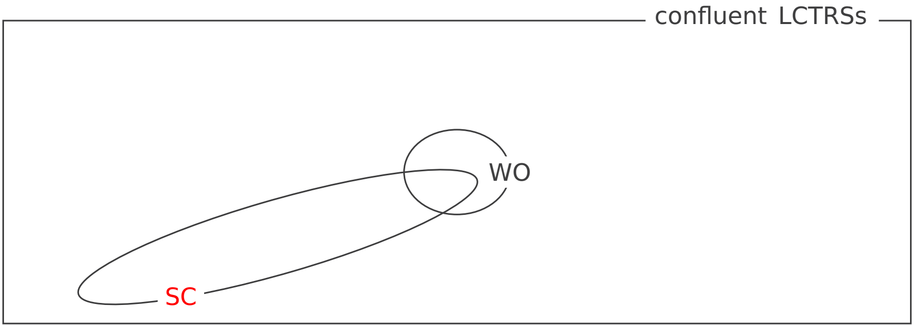
which are almost development closed:

$$\begin{aligned} & \mathsf{h}(\mathsf{g}(y, 2\cdot 2)) \approx \mathsf{c}(4,x) \quad [\varphi] \\ \overset{\sim}{\multimap}_{\geqslant 1} \; & \mathsf{g}(4,2) \approx \mathsf{c}(4,x) \quad [x = 2] \\ \overset{\sim}{\to}_{\geqslant 2} \; & \mathsf{g}(4,2) \approx \mathsf{g}(4,2) \quad [\mathsf{true}] \end{aligned}$$

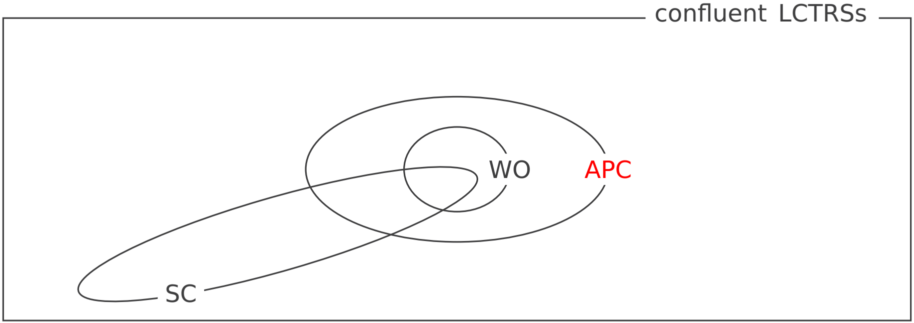
$$\begin{aligned} & \mathsf{c}(4,x) \approx \mathsf{h}(\mathsf{g}(y, 2\cdot 2)) \quad [\varphi] \\ \overset{\sim}{\multimap}_{\geqslant 1} \; & \mathsf{g}(4,2) \approx \mathsf{h}(\mathsf{g}(y, 2\cdot 2)) \quad [y = 2] \\ \overset{\sim}{\to}^{*}_{\geqslant 2} \; & \mathsf{g}(4,2) \approx \mathsf{g}(4,2) \quad [\mathsf{true}] \end{aligned}$$


confluent LCTRSs
WO



Theorem

- linear strongly closed LCTRSs are confluent — CADE 2023



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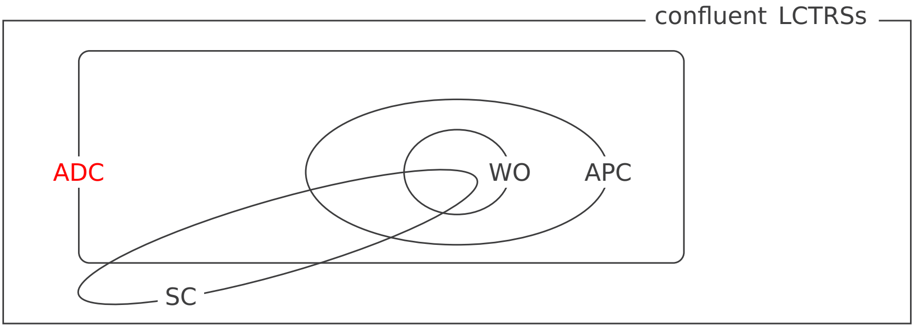

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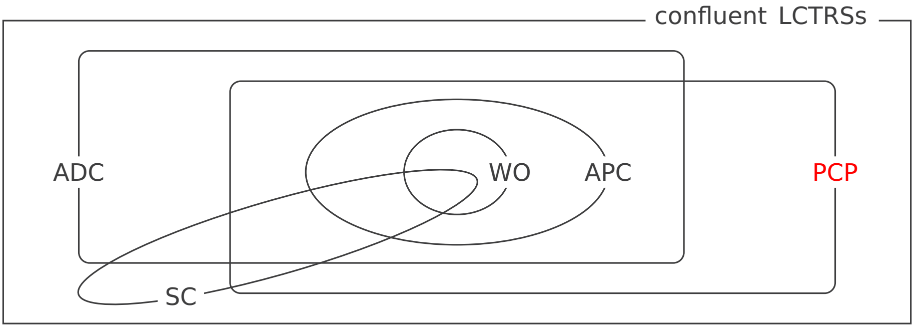

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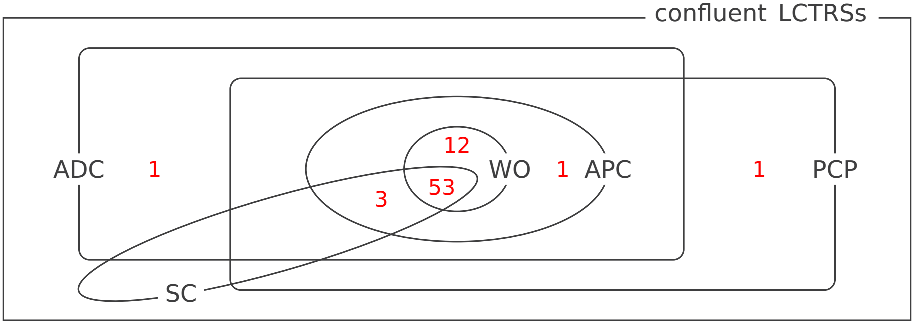
103 LCTRSs in ARI database



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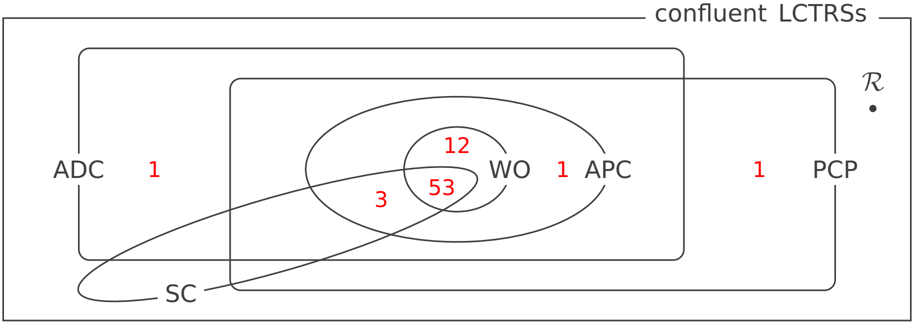

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Example

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$$\mathsf{f}(x) \to \mathsf{g}(x) \qquad \mathsf{g}(x) \to \mathsf{h}(2) \quad [x = 2z]$$

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$$\mathsf{g}(x) \approx \mathsf{h}(x) \quad [1 \leqslant x \leqslant 2]$$

  of $\mathcal{R}$ is not almost development closed

Outline

1. IWC
2. Logically Constrained Rewrite Systems
3. Critical Pairs
4. Undecidability
5. Transformation
6. Confluence Results
7. **Final Remarks**

Confluence Methods for TRSs

- critical pair closing systems
- decreasing diagrams
- development closed critical pairs
- discrimination pairs
- joinable critical pairs for terminating systems
- orthogonality
- parallel closed critical pairs
- parallel critical pairs
- redundant rules
- rule labeling
- simultaneous critical pairs
- source labeling
- strongly closed critical pairs
- tree automata
- weak orthogonality
- Z property
- · · ·

Confluence Methods for LCTRSs

development closed critical pairs

joinable critical pairs for terminating systems

orthogonality

parallel closed critical pairs

parallel critical pairs

strongly closed critical pairs

weak orthogonality

Remarks

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IWC Quiz ❶

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14th International School on Rewriting (ISR 2024)

August 25 — September 1

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